

MANY-PARAMETER M-COMPLEMENTARY GOLAY SEQUENCES AND TRANSFORMS

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Abstract

In this paper, we develop the family of Golay–Rudin–Shapiro (GRS) m -complementary many-parameter sequences and many-parameter Golay transforms. The approach is based on a new generalized iteration generating construction, associated with n unitary many-parameter transforms and n arbitrary groups of given fixed order. We are going to use multi-parameter Golay transform in Intelligent-OFDM-TCS instead of discrete Fourier transform in order to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

Keywords: complementary sequences, many-parameter orthogonal transforms, fast algorithms, OFDM systems.

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Introduction

Binary ± 1 -valued *Golay – Rudin – Shapiro* sequences (2-GRSS) associated with the cyclic group \mathbf{Z}_n^2 were introduced independently by Golay [1, 2, 3] in 1949-1951, Shapiro [4, 5] and Rudin [6] in 1951. M.J.E. Golay [2] introduced the general concept of "complementary pairs" of finite sequences all of whose entries are ± 1 . For building the classical FGRST in bases of classical 2-GRSS the following actors are used: 1) Abelian group \mathbf{Z}_2 , 2) 2-point Fourier transform \mathcal{F}_2 , and 3) complex field \mathbf{C} , *i.e.*, these transforms are associated with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$.

In previous papers [7, 8], we have shown a new unified approach to the $\mathbf{GF}(p)$ -, or Clifford-valued complementary sequences and Golay transforms. It was associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with triples

$$(\mathbf{Z}_2, \{CS_2^1(\varphi_1, \alpha_1, \gamma_1), CS_2^2(\varphi_2, \alpha_2, \gamma_2), \dots,$$

$$CS_n^1(\varphi_n, \alpha_n, \gamma_n)\}, \mathcal{Alg})$$

and $(\mathbf{Z}_2, CS_2(\varphi, \alpha, \gamma), \mathcal{Alg})$, where $\{CS_2^1(\varphi_1, \alpha_1, \gamma_1), CS_2^2(\varphi_2, \alpha_2, \gamma_2), \dots, CS_n^1(\varphi_n, \alpha_n, \gamma_n)\}$ is a set of arbitrary unitary(2x2) -transforms of type

$$CS_2(\varphi_k, \alpha_k, \gamma_k) = \begin{bmatrix} e^{i\alpha_k} \cos \varphi_k & e^{i\gamma_k} \sin \varphi_k \\ e^{-i\gamma_k} \sin \varphi_k & -e^{-i\alpha_k} \cos \varphi_k \end{bmatrix},$$

$$k = 1, \dots, n,$$

and $CS_2(\varphi, \alpha, \gamma)$ is a single transform, \mathcal{Alg} is an algebra (for example, Clifford algebra).

In this work, we develop a new unified approach to the so-called generalized multi-parameter m -complementary sequences. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with

- 1) $(\mathbf{Z}_m, \mathbf{U}_m, \mathcal{Alg})$, 2) $(\mathbf{Z}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}, \mathcal{Alg})$,
- 3) $(\mathbf{Gr}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}, \mathcal{Alg})$,
- 4) $(\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, \dots, \mathbf{Gr}_m^n\}, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}, \mathcal{Alg})$.

where $\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, \dots, \mathbf{Gr}_m^n\}$ is a set of arbitrary finite groups of given order m Here $\{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}$ is a set of arbitrary unitary ($m \times m$) – transforms represented in the many-parameter Jacobi-Euler form [9 – 10]:

$$\mathbf{U}_m^1 = \mathbf{U}_m^1(\varphi_0^1, \varphi_1^1, \dots, \varphi_q^1) = \mathbf{U}_m^1(\boldsymbol{\varphi}_q^1) = \prod_{r=1}^{m-1} \prod_{s=r+1}^m \mathbf{J}(\varphi_{r,s}^1),$$

$$\mathbf{U}_m^2 = \mathbf{U}_m^2(\varphi_0^2, \varphi_1^2, \dots, \varphi_q^2) = \mathbf{U}_m^2(\boldsymbol{\varphi}_q^2) = \prod_{r=1}^{m-1} \prod_{s=r+1}^m \mathbf{J}(\varphi_{r,s}^2),$$

....

$$\mathbf{U}_m^n = \mathbf{U}_m^n(\varphi_0^n, \varphi_1^n, \dots, \varphi_q^n) = \mathbf{U}_m^n(\boldsymbol{\varphi}_q^n) = \prod_{r=1}^{m-1} \prod_{s=r+1}^m \mathbf{J}(\varphi_{r,s}^n),$$

where

$$\mathbf{J}(\varphi_{r,s}) = \begin{matrix} & & & r & & & s & & & \\ \begin{matrix} 1 & \dots & & 0 & \dots & & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots & & & \vdots & & \vdots \\ 0 & \dots & c(\varphi_{r,s}) & \dots & s(\varphi_{r,s}) & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & s(\varphi_{r,s}) & \dots & -c(\varphi_{r,s}) & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{matrix} & , \end{matrix}$$

is the Jacobi orthonormal rotation with reflection, $\boldsymbol{\varphi}_q^1 = (\varphi_0^1, \varphi_1^1, \dots, \varphi_q^1), \dots, \boldsymbol{\varphi}_q^n = (\varphi_0^n, \varphi_1^n, \dots, \varphi_q^n)$ are the Jacobi parameters, $q = C_m^2 = m(m-1)/2$, $c(\varphi_{r,s}) = \cos(\varphi_{r,s})$, $s(\varphi_{r,s}) = \sin(\varphi_{r,s})$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay – Rudin – Shapiro* m -ary sequences) is described. In Section 3 we propose method based on new generalized iteration rule with n unitary ($m \times m$)-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n$ and single group \mathbf{Z}_m . Then we generalize the previously method on n unitary ($m \times m$)-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n$ and on n finite groups $\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, \dots, \mathbf{Gr}_m^n\}$. In Section 5 we derive fast algorithms for binary Golay transforms.

The object of the study.

New iteration construction for original Golay sequences

We begin by describing the original Golay m -complementary sequences.

Definition 1. A generalization of the Golay complementary pair, known as the Golay m -Complementary m -element Set (m -GCS) of complex-valued sequences [11]

$$m\text{-GCS} = \begin{cases} \text{com}_0(t) := (c_0(0), c_0(1), \dots, c_0(m-1)), \\ \text{com}_1(t) := (c_1(0), c_1(1), \dots, c_1(m-1)), \\ \dots, \dots, \dots \\ \text{com}_{m-1}(t) := (c_{m-1}(0), c_{m-1}(1), \dots, c_{m-1}(m-1)) \end{cases}$$

is defined by $\sum_{k=0}^{m-1} \text{COR}_k(\tau) = m \cdot \delta(\tau)$, $\sum_{k=0}^{m-1} |\text{COM}_k(z)|^2 = m$,

where $\{\text{COR}_k(\tau)\}_{k=0}^{m-1}$ are the periodic autocorrelation functions of $\{\text{com}_k(t)\}_{k=0}^{m-1}$ and $\text{COM}_k(z) = \mathcal{Z}\{\text{com}_k(t)\}$ are their \mathcal{Z} -transforms.

We use two symbols $\alpha_n \in [0, m^{n-1}-1] = \mathbf{Z}_m^n$ and $\mathbf{t}_n \in [0, m^{n-1}-1] = \mathbf{Z}_m^n$ for numeration of Golay sequences and discrete time, respectively. For integer $\alpha_n \in [0, m^{n-1}-1] = \mathbf{Z}_m^n$ and $\mathbf{t}_n \in [0, m^{n-1}-1] = \mathbf{Z}_m^n$ we shall use m -arycodes $\vec{\alpha}_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\vec{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$, where $\alpha_i, t_i \in \{0, 1, \dots, m-1\} = \mathbf{Z}_m$, $i = 1, 2, \dots, n$.

Let $\vec{\alpha}_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\vec{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$ be m -ary codes, then define

$$\alpha_n = |\vec{\alpha}_n| = \sum_{i=1}^n \alpha_{n-i+1} m^{i-1}, \text{ and } \mathbf{t}_n = |\vec{\mathbf{t}}_n| = \sum_{i=1}^n t_{n-i+1} m^{n-i}$$

as integers whose m -ary codes are $\vec{\alpha}_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\vec{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$, where α_n, t_1 are less significant bits (LSB) and α_1, t_n are most significant bits (MSB) of $\vec{\alpha}_n = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\vec{\mathbf{t}}_n = (t_1, t_2, \dots, t_n)$, respectively. Obviously,

$$\begin{aligned} \vec{\alpha}_1 &= (\alpha_1) \in \mathbf{Z}_m, & \alpha_1 &= \alpha_1 \in \mathbf{Z}_m, \\ \vec{\alpha}_2 &= (\vec{\alpha}_1, \alpha_2) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, & (\alpha_1, \alpha_2) &\in \mathbf{Z}_m \times \mathbf{Z}_m, \\ \vec{\alpha}_3 &= (\vec{\alpha}_2, \alpha_3) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, & (\alpha_2, \alpha_3) &\in \mathbf{Z}_m^2 \times \mathbf{Z}_m, \\ & \dots, \dots, \dots, & & \\ \vec{\alpha}_n &= (\vec{\alpha}_{n-1}, \alpha_n) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, & (\alpha_{n-1}, \alpha_n) &\in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m; \\ \vec{\mathbf{t}}_1 &= (t_1) \in \mathbf{Z}_m, & \mathbf{t}_1 &= t_1 \in \mathbf{Z}_m, \\ \vec{\mathbf{t}}_2 &= (\vec{\mathbf{t}}_1, t_2) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, & (\mathbf{t}_1, t_2) &\in \mathbf{Z}_m \times \mathbf{Z}_m, \\ \vec{\mathbf{t}}_3 &= (\vec{\mathbf{t}}_2, t_3) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, & (\mathbf{t}_2, t_3) &\in \mathbf{Z}_m^2 \times \mathbf{Z}_m, \\ & \dots, \dots, \dots, & & \\ \vec{\mathbf{t}}_n &= (\vec{\mathbf{t}}_{n-1}, t_n) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, & (\mathbf{t}_{n-1}, t_n) &\in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m. \end{aligned}$$

Let $\{\text{com}_{\alpha_{n+1}}^{[n+1]}(\mathbf{t}_{n+1})\}$ be m^{n+1} -element set of m complementary sequences (of length m^{n+1}), where $\alpha_{n+1}, \mathbf{t}_{n+1} = 0, 1, \dots, m^{n+1}-1$. They form rows of a $(m^{n+1} \times m^{n+1})$ -matrix $\mathbf{G}_{m^{n+1}}^{[n+1]} = [\text{com}_{\alpha_{n+1}}^{[n+1]}(\mathbf{t}_{n+1})]_{\alpha_{n+1}, \mathbf{t}_{n+1}=0}^{m^{n+1}-1}$, that is called the m -Golay matrix. Here index $[n+1]$ shows that Golay matrix have been obtained on the $n+1$ iteration step. We are going to group these rows (sequences) as

$$\mathbf{G}_{m^{n+1}}^{[n+1]} = \bigoplus_{\alpha_{n+1}=0}^{m^{n+1}-1} \text{com}_{\alpha_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}) = \bigoplus_{\alpha_n=0}^{m^n-1} \left(\bigoplus_{\alpha_{n+1}=0}^{m-1} \text{com}_{(\alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) \right) = \bigoplus_{\alpha_n=0}^{m^n-1} \begin{bmatrix} \text{com}_{(\alpha_n, 0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\alpha_n, 1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_n, m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}. \tag{1}$$

Let us to select the more fine structure of the m -Golay matrix:

$$\mathbf{G}_{m^{n+1}}^{[n+1]} = \bigoplus_{\alpha_{n+1}=0}^{m^{n+1}-1} \text{com}_{\alpha_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}) = \bigoplus_{\alpha_n=0}^{m^n-1} \begin{bmatrix} \text{com}_{(\alpha_n, 0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\alpha_n, 1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_n, m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} = \bigoplus_{\alpha_{n-1}=0}^{m^{n-1}-1} \left(\bigoplus_{\alpha_n=0}^{m-1} \begin{bmatrix} \text{com}_{(\alpha_{n-1}, \alpha_n, 0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\alpha_{n-1}, \alpha_n, 1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_{n-1}, \alpha_n, m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix} \right) = \bigoplus_{\alpha_{n-1}=0}^{m^{n-1}-1} \begin{bmatrix} \text{com}_{(\alpha_{n-1}, 0, 0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\alpha_{n-1}, 0, 1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_{n-1}, 0, m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \hline \text{com}_{(\alpha_{n-1}, 1, 0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\alpha_{n-1}, 1, 1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_{n-1}, 1, m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \vdots \\ \hline \text{com}_{(\alpha_{n-1}, m-1, 0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \text{com}_{(\alpha_{n-1}, m-1, 1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_{n-1}, m-1, m-1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}. \tag{2}$$

Example 1. For $n=1$ and $n=2$ we have, respectively,

$$\mathbf{G}_{3^t}^{[1]} = \left[\text{com}_{a_i}^{[1]}(\mathbf{t}_1) \right]_{a_i, t_i=0}^2 = \bigoplus_{a_i=0}^2 \text{com}_{a_i}^{[1]}(\mathbf{t}_1) = \begin{bmatrix} \text{com}_{(0)}^{[1]}(\mathbf{t}_1) \\ \text{com}_{(1)}^{[1]}(\mathbf{t}_1) \\ \text{com}_{(2)}^{[1]}(\mathbf{t}_1) \end{bmatrix},$$

$$\mathbf{G}_{3^2}^{[2]} = \bigoplus_{a_i=0}^2 \begin{bmatrix} \text{com}_{(a_i,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(a_i,1)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(a_i,2)}^{[2]}(\mathbf{t}_2) \end{bmatrix} = \begin{bmatrix} \text{com}_{(0,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(0,1)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(0,2)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,1)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,2)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(2,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(2,1)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(2,2)}^{[2]}(\mathbf{t}_2) \end{bmatrix}. \quad \square$$

The matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}$ is constructed by an iteration construction. The initial matrix $\mathbf{G}_m^{[1]}$ is formed by starting with an arbitrary unitary $(m \times m)$ -matrix (in many-parameter form or not)

$$\mathbf{U}_m = [A_\alpha(t)] := \mathbf{G}_m^{[1]} = \begin{bmatrix} \text{com}_0^{[1]}(\mathbf{t}_1) \\ \text{com}_1^{[1]}(\mathbf{t}_1) \\ \dots \\ \text{com}_{m-1}^{[1]}(\mathbf{t}_1) \end{bmatrix} =$$

$$= \begin{bmatrix} A_0(0) & A_0(1) & A_0(2) & \dots & A_0(m-1) \\ A_1(0) & A_1(1) & A_1(2) & \dots & A_1(m-1) \\ A_2(0) & A_2(1) & A_2(2) & \dots & A_2(m-1) \\ \dots & \dots & \dots & \dots & \dots \\ A_{m-1}(0) & A_{m-1}(1) & A_{m-1}(2) & \dots & A_{m-1}(m-1) \end{bmatrix},$$

where $A_\alpha(t) \in \mathcal{Alg}$,

$$\text{com}_\alpha^{[1]}(t) = (A_\alpha(0), A_\alpha(1), \dots, A_\alpha(m-1)).$$

Example 2. The initial matrix $\mathbf{G}_m^{[1]}$ can be the Fourier transform on Abelian group \mathbf{Z}_m :

$$\mathbf{G}_m^{[1]} = \begin{bmatrix} \text{com}_0^{[1]}(\mathbf{t}_1) \\ \text{com}_1^{[1]}(\mathbf{t}_1) \\ \text{com}_2^{[1]}(\mathbf{t}_1) \\ \dots \\ \text{com}_{m-1}^{[1]}(\mathbf{t}_1) \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \varepsilon^{1-1} & \varepsilon^{1-2} & \dots & \varepsilon^{1-(m-1)} \\ 1 & \varepsilon^{2-1} & \varepsilon^{2-2} & \dots & \varepsilon^{2-(m-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \varepsilon^{(m-1)-1} & \varepsilon^{(m-1)-2} & \dots & \varepsilon^{(m-1)-(m-1)} \end{bmatrix}, \quad (3)$$

where $\varepsilon_m = \sqrt[m]{1} \in \mathcal{Alg}$, $\text{com}_k^{[1]}(\mathbf{t}) = (1, \varepsilon^{k-1}, \varepsilon^{k-2}, \dots, \varepsilon^{k-(m-1)})$, ($k=0, 1, \dots, m-1$) are characters \mathbf{Z}_m . \square

It is easy to check that

$$\left(|\text{COM}_0(z)|^2 + |\text{COM}_1(z)|^2 + \dots + |\text{COM}_{m-1}(z)|^2 \right)_{|z|=1} = m.$$

Indeed,

$$\sum_{k=1}^{m-1} |\text{COM}_k(z)|^2 = \sum_{k=1}^{m-1} \text{COM}_k(z) \overline{\text{COM}_k(z)} =$$

$$= \sum_{k=1}^{m-1} \left(\sum_{t=0}^{m-1} a_k(t) z^t \right) \left(\sum_{s=0}^{m-1} \bar{a}_k(s) \bar{z}^s \right) =$$

$$= \sum_{s=0}^{m-1} \sum_{t=0}^{m-1} \left(\sum_{k=0}^{m-1} a_k(t) \bar{a}_k(s) \right) z^t \bar{z}^s = \sum_{s=0}^{m-1} \sum_{t=0}^{m-1} \delta_{t-s} z^t \bar{z}^s = \sum_{t=0}^{m-1} |z|^{2t},$$

since $\sum_{k=0}^{m-1} a_k(t) \bar{a}_k(s) = \delta_{t-s}$ is true for an arbitrary unitary (orthogonal) matrix. Hence,

$$\left(\sum_{k=1}^{m-1} |\text{COM}_k(z)|^2 \right)_{|z|=1} = \left(\sum_{t=0}^{m-1} |z|^{2t} \right)_{|z|=1} = m$$

and initial sequences in the form of rows of an unitary matrix (in particular case, in the form of characters $\text{com}_k(\mathbf{t}_1) = (1, \varepsilon^{k-1}, \varepsilon^{k-2}, \dots, \varepsilon^{k-(m-1)})$ of cyclic group \mathbf{Z}_m) are the Golay m -complementary sequences.

Methods

The matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}$ is constructed by an iteration construction

$$\mathbf{G}_m^{[1]}(\mathbf{U}_m^1) \xrightarrow{\mathbf{U}_m^2} \mathbf{G}_m^{[2]}(\mathbf{U}_m^1, \mathbf{U}_m^2) \xrightarrow{\mathbf{U}_m^3} \mathbf{G}_m^{[3]}(\mathbf{U}_m^1, \mathbf{U}_m^2, \mathbf{U}_m^3) \xrightarrow{\mathbf{U}_m^{n+1}} \mathbf{G}_{m^{n+1}}^{[n+1]}(\mathbf{U}_m^1, \dots, \mathbf{U}_m^n, \mathbf{U}_m^{n+1}), \quad (4)$$

where

$$\mathcal{U}_{n+1} := \{ \mathbf{U}_m^1, \dots, \mathbf{U}_m^n, \mathbf{U}_m^{n+1} \} = \{ \mathcal{U}_n, \mathbf{U}_m^{n+1} \},$$

$$\mathcal{U}_n := \{ \mathbf{U}_m^1, \dots, \mathbf{U}_m^n \}.$$

Here $\mathbf{U}_m^s(\varphi_q^s) = [A_\alpha^s(t | \varphi_q^s)]_{\alpha, t=0}^{m-1} \in SU(\mathcal{Alg}, m)$ ($s=1, 2, \dots, n$) are a sequence of unitary many-parameter $(m \times m)$ -transforms, belonging to the special unitary group $SU(\mathcal{Alg}, m)$, where $s=1, 2, \dots, n+1$ and $A_\alpha^s(t | \varphi_q^s)$ are \mathcal{Alg} -valued many-parameter sequences.

Let us assume that we have m -Golay matrix $\mathbf{G}_{m^n}^{[n]}(\mathbf{U}_1, \dots, \mathbf{U}_n) = \mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ (depending on n previous transforms $\mathbf{U}_1, \dots, \mathbf{U}_n$). We need to construct the next m -Golay matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathbf{U}_1, \dots, \mathbf{U}_n, \mathbf{U}_m^{n+1}) = \mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1})$ using only $\mathbf{G}_{m^n}^{[n]}(\mathbf{U}_1, \dots, \mathbf{U}_n)$ and \mathbf{U}_m^{n+1} . We are going to use for m -Golay matrix $\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ the same structure as in (1):

$$\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n) = \bigoplus_{a_n=0}^{m^n-1} \text{com}_{(a_n)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) =$$

$$= \bigoplus_{a_n=0}^{m^n-1} \begin{bmatrix} \text{com}_{(a_n-1,0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \text{com}_{(a_n-1,1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \dots \\ \text{com}_{(a_n-1,m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \end{bmatrix}. \quad (5)$$

For constructing $\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1})$ from $\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ we take each complementary set in the form

$$m\text{-GCS}^{[n]}(\mathcal{U}_n) = \begin{bmatrix} \text{com}_{(\alpha_{n-1},0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \text{com}_{(\alpha_{n-1},1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \dots \\ \text{com}_{(\alpha_{n-1},m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \end{bmatrix}$$

and construct m shifted versa of their components

$$m\text{-GCS}^{[n]}(\mathcal{U}_n) \begin{matrix} \nearrow \\ \rightarrow \\ \searrow \end{matrix} \begin{matrix} m\text{-GCS}_{\alpha_n=0}^{[n]}(\mathcal{U}_{n+1}), \\ m\text{-GCS}_{\alpha_n=1}^{[n]}(\mathcal{U}_{n+1}), \\ \dots \\ m\text{-GCS}_{\alpha_n=m-1}^{[n]}(\mathcal{U}_{n+1}), \end{matrix}$$

where

$$m\text{-GCS}_{\alpha_n}^{[n]}(\mathcal{U}_{n+1}) = \mathbf{U}_m^{n+1} \begin{pmatrix} \left[\begin{matrix} \mathbf{I}_\alpha \\ \mathbf{T}_\alpha^{1-m^n} \\ \dots \\ \mathbf{T}_\alpha^{(m-1)\cdot m^n} \end{matrix} \right] \tilde{\mathbf{P}}_m^{\alpha_n} \end{pmatrix} \times \begin{bmatrix} \text{com}_{(\alpha_{n-1},0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \text{com}_{(\alpha_{n-1},1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \dots \\ \text{com}_{(\alpha_{n-1},m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \end{bmatrix} \equiv \begin{bmatrix} \text{com}_{(\alpha_n,0)}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) \\ \text{com}_{(\alpha_n,1)}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) \end{bmatrix} \quad (6)$$

Here $\alpha_n = 0, 1, \dots, m-1$, $\mathbf{P}_m^{\alpha_n}$ is the cyclic permutation operator on α_n positions (modulo m), $\mathbf{T}_\alpha^{m^n s}$ is the shift operator on $m^n s$ positions $\mathbf{T}_\alpha^{m^n s} f(\mathbf{t}_n) := f(\mathbf{t}_n + m^n s)$, $\tilde{\mathbf{P}}_m$ is transposed matrix of \mathbf{P}_m .

According to (1) we obtain

$$\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1}) = \begin{matrix} \begin{matrix} \square & \square \\ \square & \square \end{matrix} \\ \alpha_n=0 \end{matrix} \begin{bmatrix} \text{com}_{(\alpha_n,0)}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) \\ \text{com}_{(\alpha_n,1)}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) \\ \dots \\ \text{com}_{(\alpha_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) \end{bmatrix} = \begin{matrix} \begin{matrix} \square & \square \\ \square & \square \end{matrix} \\ \alpha_n=0 \end{matrix} \cdot \left(\begin{matrix} \mathbf{U}_m^{n+1} \\ \mathbf{P}_m^{\alpha_n} \end{matrix} \cdot \begin{bmatrix} \mathbf{I}_\alpha \\ \mathbf{T}_\alpha^{1-m^n} \\ \dots \\ \mathbf{T}_\alpha^{(m-1)\cdot m^n} \end{bmatrix} \cdot \tilde{\mathbf{P}}_m^{\alpha_n} \right) \times \begin{bmatrix} \text{com}_{(\alpha_{n-1},0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \text{com}_{(\alpha_{n-1},1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \\ \dots \\ \text{com}_{(\alpha_{n-1},m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) \end{bmatrix}$$

and, consequently,

$$\text{com}_{(\alpha_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) = \sum_{\beta_n=0}^{m-1} a_{\alpha_{n+1}}^{n+1}(\beta_n) \mathbf{T}_\alpha^{m^n(\beta_n \oplus \alpha_n)} \text{com}_{(\alpha_{n-1}, \beta_n)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n).$$

Since $\mathbf{t}_{n+1} = (\mathbf{t}_n, t_{n+1})$, then believing $t_{n+1} = \alpha_n \oplus_m \beta_n$, we obtain:

$$\begin{aligned} \text{com}_{(\alpha_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1}) &= \text{com}_{(\alpha_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1} | \mathcal{U}_{n+1}) = \\ &= \sum_{t_{n+1}=0}^{m-1} A_{\alpha_{n+1}}^{n+1}(\alpha_n \oplus_m t_{n+1}) \mathbf{T}_\alpha^{m^n t_{n+1}} \text{com}_{(\alpha_{n-1}, \alpha_n \oplus_m t_{n+1})}^{[n]}(\mathbf{t}_n | \mathcal{U}_n) = \\ &= \sum_{t_{n+1}=0}^{m-1} A_{\alpha_{n+1}}^{n+1}(\alpha_n \oplus_m t_{n+1}) \mathbf{T}_\alpha^{m^n t_{n+1}} \text{com}_{(\alpha_{n-1}, \alpha_n \oplus_m t_{n+1})}^{[n]}(\mathbf{t}_n + m^n t_{n+1} | \mathcal{U}_n). \end{aligned} \quad (8)$$

So,

$$\begin{aligned} \text{com}_{(\alpha_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1} | \mathcal{U}_{n+1}) &= \\ &= A_{\alpha_{n+1}}^{n+1}(\alpha_n \oplus_m t_{n+1}) \cdot \text{com}_{(\alpha_{n-1}, \alpha_n \oplus_m t_{n+1})}^{[n]}(\mathbf{t}_n | \mathcal{U}_n). \end{aligned} \quad (9)$$

It is finally recurrent relation between m -complementary sequences of $\mathbf{G}_{m^{n+1}}^{[n+1]}[\mathcal{U}_{n+1}]$ and $\mathbf{G}_m^{[n]}[\mathcal{U}_n]$.

From (9) we obtain expression for $\text{com}_{\alpha_{n+1}}^{[n+1]}(\mathbf{t}_{n+1} | \mathcal{U}_{n+1})$:

$$\text{com}_{\alpha_{n+1}}^{[n+1]}(\mathbf{t}_{n+1}) = \prod_{s=1}^n A_{\alpha_{s+1} \oplus t_{s+2}}^{s+1}(\alpha_s \oplus_m t_{s+1}), \quad \alpha_0, t_{n+2} \equiv 0. \quad (10)$$

In particular, for matrices in the form of the Fourier transform $\mathbf{U}_m^1 = \mathbf{U}_m^2 = \dots = \mathbf{U}_m^n = [\varepsilon_m^{\alpha t}]$ we have

$$\begin{aligned} \text{com}_{(\alpha_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) &= \text{com}_{(\alpha_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) = \\ &= \varepsilon_m^{s=1}(\alpha_s \oplus_m t_{s+1})(\alpha_{s+1} \oplus_m t_{s+2}). \end{aligned} \quad (11)$$

Where $\alpha_0, t_{n+2} \equiv 0$. New sequences in (9) are orthogonal and m -complementary sequences.

Generalizations

In this section, we introduce generalized m -complementary sequences. It is based on using new permutation matrices $\mathbf{P}_m^{\alpha_n}$ in (7). The mappings $g: \mathbf{X} \rightarrow \mathbf{X}$ of a set \mathbf{X} into (or onto) itself are of particular importance. They form the following set $\mathbf{X}^{\mathbf{X}} = \{g|g: \mathbf{X} \rightarrow \mathbf{X}\}$.

Definition 2. One-to-one map from a set \mathbf{X} to itself $g: \mathbf{X} \rightarrow \mathbf{X}$, $x' = g(x) = g \circ x$ is called a transformation of the set \mathbf{X} .

If \mathbf{X} is finite and consists of m elements (for example, $\mathbf{X} = \{0, 1, 2, \dots, m\}$) then a transformation of the set \mathbf{X} is called a *permutation*. As is well known, the set of all permutations of \mathbf{X} forms a group $S_m = \text{Sum}\{\mathbf{X}\}$ in which the product $\sigma\pi$ of a pair of permutations σ, π is defined by $(\sigma\pi) \circ x := \sigma \circ (\pi \circ x)$.

If \mathbf{X} contains more than two elements, S_m is not commutative. Any subgroup of S_m is called a *permutation group* on \mathbf{X} , or a *group of permutations* of \mathbf{X} . We shall say that the permutations in $\text{Sym}(\mathbf{X})$ *act* or *operate* on the elements of \mathbf{X} .

Definition 3. A homomorphism of a group on a set $h: \mathbf{Gr} \rightarrow \text{Sym}\{\mathbf{X}\}$ is called a *permutation representation* (or *realization*) of.

The image $h(\mathbf{Gr}) \subset \text{Sym}\{\mathbf{X}\}$ is a permutation group and the elements of are represented as permutations of . A permutation representation is equivalent to an action of on the set : To specify an action, we need to define for element $g \in \mathbf{Gr}$ the corresponding permutation $h(g)$ of , that is, $h(g) \circ x$ for any $x \in \mathbf{X}$. We are going to write $h(g) \circ x$

in the short form $g \circ x$ and to call the group of transformations of \mathcal{X} . The pair (\mathcal{X}, G) is called a space with transformation group the elements $x \in \mathcal{X}$ are called points of the space.

Definition 4. If G is a permutation group of degree m , then the permutation representation of G is the linear permutation representation of $G: \mathbf{P}: \mathbf{Gr} \rightarrow \text{GL}_m(\text{Alg})$ which maps G to the corresponding permutation matrix $\mathbf{P}(g)$.

That is, acts on \mathcal{X} by permuting the standard basis vectors $\{e_n\}_{n \in \mathcal{X}} \in \text{Alg}^m$ such that

$$\mathbf{P}(g)e_n = e_{g \circ n} = e_{n'}, \quad n' \in \{e_n\}_{n \in \mathcal{X}},$$

where $\mathbf{P}(g)$'s are the operators in Alg^m which define the above mentioned linear representation.

Example 3. Let

$$\mathbf{X} = [0, 1, \dots, m-1], \quad \mathbf{Gr} = \mathbf{Z}_m = \langle \{0, 1, \dots, m-1\}, \oplus_m \rangle$$

be the cyclic group of order m . Then

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad \mathbf{P}(1) = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & \ddots \\ 1 & & & \end{bmatrix}, \quad \mathbf{P}(2) = \begin{bmatrix} & & 1 & \\ & & & 1 \\ & & & & \ddots \\ 1 & & & & \end{bmatrix}, \dots, \quad \mathbf{P}(m-1) = \begin{bmatrix} & & & & 1 \\ & & & & & 1 \\ & & & & & & \ddots \\ & & & & & & & 1 \\ 1 & & & & & & & \end{bmatrix}.$$

In particular, for $m=2$ and $m=3$ we have

$$\mathbf{P}(0) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad \mathbf{P}(1) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix};$$

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad \mathbf{P}(1) = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}, \quad \mathbf{P}(2) = \begin{bmatrix} & & 1 \\ 1 & & \\ & & 1 \end{bmatrix}. \quad \square$$

In expression (7) was used linear permutation representation $\mathbf{P}(g)$ of only one group G . However, we can use others finite groups of given order m . Let $\mathbf{Gr} = \mathbf{Gr}_m = \{g_\alpha\}_{\alpha=0}^{m-1}$ be a group of given order m and $\{\mathbf{P}(g_\alpha)\}_{\alpha=0}^{m-1}$. Then

$$\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1}; \mathbf{Gr}_m) = \begin{bmatrix} \text{com}_{(a_n, 0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathbf{Gr}_m) \\ \text{com}_{(a_n, 1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathbf{Gr}_m) \\ \dots \\ \text{com}_{(a_n, m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathbf{Gr}_m) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{1 \cdot m^n} \\ \dots \\ \mathbf{T}_{\mathbf{t}_n}^{(m-1) \cdot m^n} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{P}_m(g_{a_n}) \\ \dots \\ \tilde{\mathbf{P}}_m(g_{a_n}) \end{bmatrix} \cdot \begin{bmatrix} \text{com}_{(a_{n-1}, 0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathbf{Gr}_m) \\ \text{com}_{(a_{n-1}, 1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathbf{Gr}_m) \\ \dots \\ \text{com}_{(a_{n-1}, m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathbf{Gr}_m) \end{bmatrix} \quad (12)$$

is the Golay matrix associated with triple $(\mathbf{Gr}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^{n+1}\}, \text{Alg})$.

Example 4. For $m=4$ we have two groups: $\mathbf{Z}_4 = \{0, 1, 2, 3\}$ and $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. For both groups we have the following permutation representations:

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \mathbf{P}(1) = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}, \quad \mathbf{P}(2) = \begin{bmatrix} & & 1 & \\ & & & 1 \\ & & & & 1 \\ 1 & & & \end{bmatrix}, \quad \mathbf{P}(3) = \begin{bmatrix} & & & 1 \\ & & & & 1 \\ & & & & & 1 \\ 1 & & & \end{bmatrix},$$

$$\mathbf{P}(0,0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad \mathbf{P}(0,1) = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}, \quad \mathbf{P}(1,0) = \begin{bmatrix} & & 1 & \\ & & & 1 \\ & & & & 1 \\ 1 & & & \end{bmatrix}, \quad \mathbf{P}(1,1) = \begin{bmatrix} & & & 1 \\ & & & & 1 \\ & & & & & 1 \\ 1 & & & \end{bmatrix}.$$

Hence, we can construct two different set of Golay matrices associated with two triples

- 1) $(\mathbf{Z}_4, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^{n+1}\}, \text{Alg})$,
- 2) $(\mathbf{Z}_2 \times \mathbf{Z}_2, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^{n+1}\}, \text{Alg})$,

respectively. \square

Let $\mathcal{G}_{n+1} := \{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, \dots, \mathbf{Gr}_m^n, \mathbf{Gr}_m^{n+1}\} = \{\mathcal{G}_m^n, \mathbf{Gr}_m^{n+1}\}$ be a set of arbitrary groups of given order $m : \mathbf{Gr}_m^1 = \{g_{\alpha_1}^1\}_{\alpha_1=0}^{m-1}, \dots, \mathbf{Gr}_m^{n+1} = \{g_{\alpha_{n+1}}^1\}_{\alpha_{n+1}=0}^{m-1}$. Then we

can use on each k^{th} iteration permutation representations $\{\mathbf{P}_m^k(g_{\alpha_k})\}_{\alpha_k=0}^{m-1}$ for \mathbf{Gr}_m^k . In this case, we obtain the following Golay transform

$$\mathbf{G}_{m^{n+1}}(\mathcal{U}_{n+1}; \mathcal{G}_{n+1}) = \begin{bmatrix} \text{com}_{(\mathbf{a}_n, 0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathcal{G}_{n+1}) \\ \text{com}_{(\mathbf{a}_n, 1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathcal{G}_{n+1}) \\ \dots \\ \text{com}_{(\mathbf{a}_n, m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_{n+1}; \mathcal{G}_{n+1}) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & & & \\ & \mathbf{T}_{\mathbf{t}_n}^{1-m^n} & & \\ & & \ddots & \\ & & & \mathbf{T}_{\mathbf{t}_n}^{(m-1) \cdot m^n} \end{bmatrix} \cdot \tilde{\mathbf{P}}_m^{n+1}(g_{\alpha_n}) \cdot \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathcal{G}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathcal{G}_n) \\ \dots \\ \text{com}_{(\mathbf{a}_{n-1}, m-1)}^{[n]}(\mathbf{t}_n | \mathcal{U}_n; \mathcal{G}_n) \end{bmatrix} \quad (13)$$

It is associated with triple

$$\left(\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, \dots, \mathbf{Gr}_m^{n+1}\}, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^{n+1}\}, \mathcal{A}l\mathcal{G} \right).$$

Fast Golay transforms

Let us consider expressions (8) and (9) for $m=2$ (i.e., expressions (6) and (7) from our work [7]):

$$\text{com}_{(\mathbf{a}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \text{com}_{(\mathbf{a}_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) = \sum_{t_{n+1}=0}^1 (-1)^{(\alpha_n \oplus t_{n+1}) \alpha_{n+1}} \text{com}_{(\mathbf{a}_{n-1}, \alpha_n \oplus t_{n+1})}^{[n]}(\mathbf{t}_n + 2^n \cdot t_{n+1}), \quad (14)$$

$$\text{com}_{(\mathbf{a}_{n-1}, \alpha_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) = (-1)^{(\alpha_n \oplus t_{n+1}) \alpha_{n+1}} \text{com}_{(\mathbf{a}_{n-1}, \alpha_n \oplus t_{n+1})}^{[n]}(\mathbf{t}_n) \times (-1)^{\alpha_n \alpha_{n+1}} (-1)^{\alpha_{n+1} t_{n+1}} \text{com}_{(\mathbf{a}_{n-1}, \alpha_n \oplus t_{n+1})}^{[n]}(\mathbf{t}_n) \quad (15)$$

and find matrix representations of these expressions. We introduce the following σ -parametrized $(2^n \times 2^n)$ -matrix:

$$\sigma \mathbf{G}_{2^n}^{[n]} := \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{cases} {}^0 \mathbf{G}_{2^n}^{[n]} = \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n) \\ 1 \end{bmatrix}, & \sigma = 0, \\ {}^1 \mathbf{G}_{2^n}^{[n]} = \begin{bmatrix} 1 \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1; \end{cases}$$

$$= \begin{cases} {}^0 \mathbf{G}_{2^n}^{[n]} = \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 0 \\ {}^1 \mathbf{G}_{2^n}^{[n]} = \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1, \end{cases}$$

and construct the direct sum of introduced matrices

$$\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \bigoplus_{\sigma=0}^1 (\sigma) \mathbf{G}_{2^n}^{[n]} = \begin{bmatrix} (0) \mathbf{G}_{2^n}^{[n]} & \\ & (1) \mathbf{G}_{2^n}^{[n]} \end{bmatrix} = \begin{bmatrix} (\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0) \mathbf{G}_{2^n}^{[n]} & \\ & (\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1) \mathbf{G}_{2^n}^{[n]} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n) \\ \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} & \\ & \begin{bmatrix} \text{com}_{(\mathbf{a}_{n-1}, 1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \text{com}_{(\mathbf{a}_{n-1}, 0)}^{[n]}(\mathbf{t}_n + 2^n) \end{bmatrix} \end{bmatrix} \quad (16)$$

From (16) we see that $\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]}$ represents $\text{com}_{(\mathbf{a}_{n-1}, \alpha_n \oplus t_{n+1})}^{[n+1]}(\mathbf{t}_n + 2^n \cdot t_{n+1})$ in (14). It is easy to see, that

$$\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \begin{bmatrix} [\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0] & \\ & [\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1] \end{bmatrix} \times \begin{bmatrix} \mathbf{G}_{2^n}^{[n]} & \\ & \mathbf{G}_{2^n}^{[n]} \end{bmatrix} = [\delta_{\alpha_{n+1}}^{(2)}(t_{n+1}) [\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{t_{n+1}}]] \times [\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]}] = \mathbf{P}_2^{[t_{n+1}]} \cdot [\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]}],$$

where

$$\mathbf{P}_{2^{n+1}}^{\{t_{n+1}\}} := \left[\delta_{\alpha_{n+1}}^{(2)}(t_{n+1}) \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{\{t_{n+1}\}} \right] \right] = \left[\begin{array}{c|c} \mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0 & \\ \hline & \mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1 \end{array} \right]$$

is the permutation matrix with controlling digit $\{t_{n+1}\}$. According to (15) the Golay matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ is the product of three matrices

$$\mathbf{G}_{2^{n+1}}^{[n+1]} = \Delta \{(-1)^{\alpha_n \alpha_{n+1}}\} \left[\delta_{\alpha_n, t_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}} \right] \tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \Delta \{(-1)^{\alpha_n \alpha_{n+1}}\} \left[\delta_{\alpha_n, t_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}} \right], \tag{17}$$

$$\left[\delta_{\alpha_{n+1}}^{(2)}(t_{n+1}) \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{\{t_{n+1}\}} \right] \right] \left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]} \right] = \Delta \{(-1)^{\alpha_n \alpha_{n+1}}\} \left[\delta_{\alpha_n, t_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}} \right] \mathbf{P}_{2^{n+1}}^{\{t_{n+1}\}} \left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]} \right].$$

Where $\Delta \{(-1)^{\alpha_n \alpha_{n+1}}\} = \text{diag} \{(-1)^{\alpha_n \alpha_{n+1}}\}$ is diagonal matrix, and $\left[\delta_{\alpha_n, t_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}} \right]$ has the following structure

$$\left[\delta_{\alpha_n, t_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}} \right] = \left[\mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] = \left[\mathbf{I}_{2^n} \middle| \mathbf{I}_{2^n} \right] \hat{\otimes} \left[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right] =$$

$$= \left[\left[\delta_{\alpha_n, t_n}^{(2^n)} \right] \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \left[\delta_{\alpha_n, t_n}^{(2^n)} \right] \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] =$$

	$t_{n+1} = 0$		$t_{n+1} = 1$	
$\alpha_{n+1} = 0$	1		1	:= $\mathbb{N}_{2^{n+1}}$.
$\alpha_{n+1} = 1$	1		-1	
$\alpha_{n+1} = 0$	1		1	
$\alpha_{n+1} = 1$	1		-1	
$\alpha_{n+1} = 0$	⋮		⋮	
$\alpha_{n+1} = 1$	⋮		⋮	
$\alpha_{n+1} = 0$	1		1	
$\alpha_{n+1} = 1$	1		-1	

Here $\hat{\otimes}$ is new tensor product:

$$\left[\mathbf{I}_{2^n} \middle| \mathbf{I}_{2^n} \right] \hat{\otimes} \left[\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right] := \left[\mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right].$$

From recurrent relation (17) we obtain

$$\mathbf{G}_{2^n}^{[n]} = \left(\prod_{k=2}^n \left[\mathbf{I}_{2^{n-k}} \otimes \Delta_{2^k} \cdot \mathbb{N}_{2^k} \cdot \mathbf{P}_{2^k}^{\{t_k\}} \right] \right) \cdot \left[\mathbf{I}_{2^{n-k+1}} \otimes \mathbf{G}_{2^1}^{[1]} \right] = \prod_{k=2}^n \left(\mathbf{I}_{2^{n-k}} \otimes \left[\Delta \{(-1)^{\alpha_{k-1} \alpha_k}\} \right] \cdot \left[\delta_{\alpha_{k-1}, t_{k-1}}^{(2^{k-1})}(-1)^{\alpha_k t_k} \right] \cdot \left[\delta_{\alpha_k}^{(2)}(t_k) \left[\mathbf{I}_{2^{k-2}} \otimes \mathbf{P}_2^{\{t_k\}} \right] \right] \right) \cdot \left[\mathbf{I}_{2^{n-k+1}} \otimes \mathbf{G}_{2^1}^{[1]} \right]. \tag{19}$$

This expression represents the fast algorithm for the Golay transform.

Example 5.

$$\mathbf{G}_{2^2}^{[2]} = \begin{bmatrix} \text{com}_{(0,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(0,1)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,0)}^{[2]}(\mathbf{t}_2) \\ \text{com}_{(1,1)}^{[2]}(\mathbf{t}_2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & -1 & \\ & & 1 & 1 \\ & & & 1 & -1 \end{bmatrix} =$$

$$= \left[\mathbf{I}_{2^0} \otimes \Delta_{2^2} \cdot \mathbb{N}_{2^2} \cdot \mathbf{P}_{2^2}^{\{t_2\}} \right] \cdot \left[\mathbf{I}_{2^1} \otimes \mathbf{G}_{2^1}^{[1]} \right].$$

$$\mathbf{G}_{2^3}^{[3]} = \begin{bmatrix} \text{com}_{(0,0,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(0,0,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(0,1,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(0,1,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,0,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,0,1)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,1,0)}^{[3]}(\mathbf{t}_3) \\ \text{com}_{(1,1,1)}^{[3]}(\mathbf{t}_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix} \times$$

$$\begin{aligned}
 & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \times \\
 & \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \times \\
 & \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \times \\
 & \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \\
 & = [\mathbf{I}_{2^0} \otimes \Delta_{2^3} \cdot \mathbb{N}_{2^3} \cdot \mathbf{P}_{2^3}^{(t_3)}] \cdot [\mathbf{I}_{2^1} \otimes \Delta_{2^2} \cdot \mathbb{N}_{2^2} \cdot \mathbf{P}_{2^2}^{(t_2)}] \cdot [\mathbf{I}_{2^2} \otimes \mathbf{G}_{2^1}^{[1]}]. \quad \square
 \end{aligned}$$

Conclusion and future researches

In this paper, we have shown a new unified approach to the so-called generalized multi-parameter m -complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with

- 1) $(\mathbf{Z}_m, \mathbf{U}_m, \mathcal{A}lg)$,
- 2) $(\mathbf{Z}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}, \mathcal{A}lg)$,
- 3) $(\mathbf{G}\mathbf{r}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}, \mathcal{A}lg)$ or with
- 4) $(\{\mathbf{G}\mathbf{r}_m^1, \mathbf{G}\mathbf{r}_m^2, \dots, \mathbf{G}\mathbf{r}_m^n\}, \{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}, \mathcal{A}lg)$,

where $\{\mathbf{U}_m^1, \mathbf{U}_m^2, \dots, \mathbf{U}_m^n\}$ is a set of arbitrary unitary $(m \times m)$ -transforms and $\{\mathbf{G}\mathbf{r}_m^1, \mathbf{G}\mathbf{r}_m^2, \dots, \mathbf{G}\mathbf{r}_m^n\}$ is a set of arbitrary groups of given order m . Furthermore, we have derived demonstrated fast algorithms for Golay transforms.

We are going to use generalized multi-parameter m -complementary sequences as subcarriers of Intelligent OFDM telecommunication system. Most of the data transmission systems nowadays use orthogonal frequency division multiplexing telecommunication system (OFDM-TCS) based on the discrete Fourier transform

(DFT) \mathcal{F}_N . The conventional OFDM will be denoted by the symbol \mathcal{F}_N -OFDM. Conventional OFDM-TCS makes use of signal orthogonality of the multiple sub-carriers $e^{j2\pi kn/N}$ (complex exponential harmonics). Sub-carriers $\{\mathbf{subc}_k(n)\}_{k=0}^{N-1} = \{e^{j2\pi kn/N}\}_{k=0}^{N-1}$ form matrix of DFT $\mathcal{F}_N = [\mathbf{subc}_k(n)]_{k,n=0}^{N-1} \equiv [e^{j2\pi kn/N}]_{k,n=0}^{N-1}$.

At the time, the idea of using the fast algorithm of different orthogonal transforms $\mathbf{U}_N = [\mathbf{subc}_k(n)]_{k,n=0}^{N-1}$ for a software-based implementation of the OFDM's modulator and demodulator, transformed this technique from an attractive, but difficult to implement idea, into an incredibly successful story of the data transmission. OFDM-TCS, based on arbitrary orthogonal (unitary) transform \mathbf{U}_N will be denoted as \mathbf{U}_N -OFDM. The idea which links \mathcal{F}_N -OFDM and \mathbf{U}_N -OFDM is that, in the same manner that the complex exponentials $\{e^{j2\pi kn/N}\}_{k=0}^{N-1}$ are orthogonal to each other, the members of a family of \mathbf{U}_N -sub-carriers $\{\mathbf{subc}_k(n)\}_{k=0}^{N-1}$ (rows of the matrix \mathbf{U}_N) will satisfy the same property. The \mathbf{U}_N -OFDM reshapes the multi-carrier transmission concept, by using carriers $\{\mathbf{subc}_k(n)\}_{k=0}^{N-1}$ in-

stead of OFDM's complex exponentials $\{e^{j2\pi kn/N}\}_{k=0}^{N-1}$. In this paper, we propose a simple and effective anti-eavesdropping and anti-jamming Intelligent OFDM system, based on MPTs. In our Intelligent-OFDM-TCS we are going to use multi-parameter Golay transform $\mathbf{G}_{2^m}(\varphi_1, \varphi_2, \dots, \varphi_q)$ at the place of DFT \mathcal{F}_N . We are going to study of Intell- $\mathbf{G}_{2^m}(\varphi_1, \varphi_2, \dots, \varphi_q)$ -OFDM-TCS to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

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